Modeling And Pricing Event Risk
Solving For Event Probability

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In many companies the risk profile includes not only the risks inherent in running a profitable business but also includes significant event risk. In this white paper we will define event risk to be the potential loss of a major customer that cannot be replaced. We will build a model the extracts the survival rate implied by the company's discounted cash flow valuation. The survival rate will allow us to calculate the event probability (i.e. jump intensity) and the go-forward weighted-average life in years of the major customer relationship. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

ABC Company has one major customer that represents 60% if its annualized revenue. If that customer leaves it is assumed that annualized revenue at that point in time would permanently decrease by 60% (i.e. that customer cannot be replaced).

Table 1: Go-Forward Model Assumptions

Symbol	Description	Value
C_0	Annualized cash flow at time zero ($)	1,000,000
μ	Cash flow secular growth rate (%)	5.00
κ	Discount rate excluding event risk (%)	12.00
ϕ	Discount rate adjustment for event risk (%)	3.00
ω	Jump size as percent of annualized cash flow (%)	60.00

Our task is to answer the following questions...

Question 1: Using the company valuation what is the implied probability that the major customer relationship will last an additional 5 years?

Question 2: What is the implied weighted-average life of this customer relationship?

Building Our Model

In Table 1 above we defined the variable C_0 to be annualized free cash flow at time zero, the variable μ to be the free cash flow annual growth rate, the variable κ to be the risk-adjusted annual discount rate excluding event risk, and the variable ϕ to be the discount rate markup to account for event risk. Using these definitions the standard DCF equation for company value at time zero is...

$$V_0 = \frac{C_0\,(1+\mu)}{\kappa+\phi-\mu} \tag{1}$$

We will define the variable n to be year number and the variable Δ to be the survival rate, which is the conditional probability that the major customer relationship as defined in the hypothetical problem above will be intact over the time interval $[n, n+1]$ given that that relationship was intact at the end of year n. The unconditional probability that the customer relationship will be intact over the time interval $[0, n]$ is therefore...

$$\mathrm{P}\Bigg[\text{Customer relationship survives over the time interval } [0,n]\Bigg] = \Delta^n \tag{2}$$

We will define the variable C_n to be expected cash flow in year n. Using Equation (2) above and the model assumptions in Table 1 above the equation for expected cash flow is...

$$\mathbb{E}\Bigg[C_n\Bigg] = C_0\,(1+\mu)^n\,\Delta^n + (1-\omega)\,C_0\,(1+\mu)^n\,(1-\Delta^n) \tag{3}$$

Using Equation (3) above the equation for the present value at time zero of expected cash flow in year n is...

$$\mathbb{E}\Bigg[C_n\,(1+\kappa)^{-n}\Bigg] = C_0\,(1+\mu)^n\,(1+\kappa)^{-n}\,\Delta^n + (1-\omega)\,C_0\,(1+\mu)^n\,(1+\kappa)^{-n}\,(1-\Delta^n) \tag{4}$$

We will define the variable θ to be the following ratio...

$$\theta = \frac{1+\mu}{1+\kappa}\ \text{...where...}\ \mu < \kappa \tag{5}$$

Using Equation (5) above we can rewrite Equation (4) above as...

$$\mathbb{E}\Bigg[C_n\,(1+\kappa)^{-n}\Bigg] = \Delta^n C_0\,\theta^n + (1-\Delta^n)\,(1-\omega)\,C_0\,\theta^n = \omega\,C_0\,(\Delta\,\theta)^n + (1-\omega)\,C_0\,\theta^n \tag{6}$$

Using Equation (6) above the equation for company value at year zero is...

$$V_0 = \sum_{n=1}^{\infty}\mathbb{E}\Bigg[C_n\,(1+\kappa)^{-n}\Bigg] = C_0\Bigg[\omega\sum_{n=1}^{\infty}(\Delta\,\theta)^n + (1-\omega)\sum_{n=1}^{\infty}\theta^n\Bigg] = C_0\Bigg[\omega\,\Delta\,\theta\sum_{n=0}^{\infty}(\Delta\,\theta)^n + (1-\omega)\,\theta\sum_{n=0}^{\infty}\theta^n\Bigg] \tag{7}$$

Note the following solutions to the infinite sums...

$$\sum_{n=0}^{\infty}(\Delta\,\theta)^n = \frac{1}{1-\Delta\,\theta}\ \text{...and...}\ \sum_{n=0}^{\infty}\theta^n = \frac{1}{1-\theta} \tag{8}$$

Using the solutions in Equation (8) above the solution to valuation Equation (7) above is...

$$V_0 = C_0\left[\frac{\omega\,\Delta\,\theta}{1-\Delta\,\theta} + \frac{(1-\omega)\,\theta}{1-\theta}\right] \tag{9}$$

Solving For The Survival Rate

We will start by equating Equations (1) and (9) above...

$$\frac{C_0\,(1+\mu)}{\kappa+\phi-\mu} = C_0\left[\frac{\omega\,\Delta\,\theta}{1-\Delta\,\theta} + \frac{(1-\omega)\,\theta}{1-\theta}\right] \tag{10}$$

Note that we can eliminate and rearrange terms such that Equation (10) above becomes...

$$\frac{\omega\,\Delta\,\theta}{1-\Delta\,\theta} = \frac{1+\mu}{\kappa+\phi-\mu} - \frac{(1-\omega)\,\theta}{1-\theta} \tag{11}$$

Note that the derivative of the left hand side of Equation (11) above with respect to the variable Δ is...

$$\frac{\delta}{\delta\,\Delta}\,\frac{\omega\,\Delta\,\theta}{1-\Delta\,\theta} = \frac{\omega\,\theta}{(1-\Delta\,\theta)^2} \tag{12}$$

We want to use Equation (11) above and solve for the variable Δ. Because the variable Δ is in the numerator and denominator of the left hand side of that equation and that quotient is nonlinear we will use the Newton Raphson method of solving nonlinear equations. Using Equations (11) and (12) above we will make the following function definitions...

$$f(\Delta\,actual) = \frac{1+\mu}{\kappa+\phi-\mu} - \frac{(1-\omega)\,\theta}{1-\theta}\ \Bigg|\ f(\Delta\,guess) = \frac{\omega\,\Delta\,\theta}{1-\Delta\,\theta}\ \Bigg|\ f'(\Delta\,guess) = \frac{\omega\,\theta}{(1-\Delta\,\theta)^2} \tag{13}$$

Noting the function definitions in Equation (13) above, to solve for Δ via the Newton Raphson method we would iterate the following equation until [new guess] = [old guess]...

$$\text{new}\Delta\ \text{guess} = f(\Delta\,actual) - \frac{f(\text{old}\ \Delta\ \text{guess})}{f'(\text{old}\ \Delta\ \text{guess})} \tag{14}$$

Jump Intensity And Customer Weighted-Average Life

We will define the variable λ to be jump intensity. A jump is defined as the realization of an event at some over the time interval $[0, t]$. From the perspective of time zero the equation for the continuous-time survival probability is... [1]

$$\text{Prob}\Bigg[\text{Surviving until time } t\Bigg] = \text{Exp}\Bigg\{ -\lambda\, t\Bigg\} \tag{15}$$

In Equation (2) above we defined the variable Δ to be the survival rate, which is the probability that the jump event will not occur over the time interval $[0, 1]$. Using Equation (15) above we can make the following statement...

$$\text{if... } \Delta = \text{Prob}\Bigg[\text{Surviving over time interval } [0,1]\Bigg] = \text{Exp}\Bigg\{ -\lambda \times 1\Bigg\} \text{ ...then... } \lambda = -\ln\Delta \tag{16}$$

Using Equation (15) above, from the perspective of time zero the equation for the probability of an event ocurring over the time interval $[t, t + \delta t]$ is...

$$\text{Prob}\Bigg[\text{Event over time interval } [t, t+\delta t]\Bigg] = -\frac{\delta}{\delta t}\,\text{Exp}\Bigg\{ -\lambda\, t\Bigg\} = \lambda\,\text{Exp}\Bigg\{ -\lambda\, t\Bigg\} \tag{17}$$

Using Equation (17) above from the perspective of time zero the equation for the probability of an event ocurring over the immediate time interval $[0, \delta t]$ is...

$$\text{Prob}\Bigg[\text{Event over time interval } [0, \delta t]\Bigg] = \lambda\,\text{Exp}\Bigg\{ -\lambda \times 0\Bigg\} = \lambda \tag{18}$$

Using the result of Equation (18) above it can be shown that the equation for weighted-average customer life is... [2]

$$\text{WAL} = \int\limits_0^\infty t\,\text{Exp}\Bigg\{ -\lambda\, t\Bigg\}\,\delta t = \frac{1}{\lambda} \tag{19}$$

The Solution To Our Hypothetical Problem

Question 1: Using the company valuation what is the implied probability that the major customer relationship will last an additional 5 years?

By interating Equation (14) above we can solve for the variable Δ as follows...

Table 2: Solving For The Survival Rate Using Newton Raphson

Iteration	guess s	f(actual s)	f(guess s)	f'(guess s)	new guess
1	1.0000	2.7182	5.67	59.31	0.9502
2	0.9502	2.7182	3.66	27.41	0.9157
3	0.9157	2.7182	2.89	18.36	0.9063
4	0.9063	2.7182	2.73	16.68	0.9058
5	0.9058	2.7182	2.72	16.60	0.9058

From the table above the survival rate is...

$$\Delta = 0.9058 \tag{20}$$

Using Equations (16) and (20) above the equation for jump intensity is...

$$\lambda = -\ln\Delta = -\ln 0.9058 = 0.0989 \tag{21}$$

Using Equations (15) and (21) above the answer to the question is...

$$\text{Prob}\Bigg[\text{Surviving until time } t = 5\Bigg] = \text{Exp}\Bigg\{ -0.0989 \times 5\Bigg\} = 0.6099 \tag{22}$$

Question 2: What is the implied weighted-average life of this customer relationship?

Using Equation (19) and (22) above the answer to the question is...

$$\text{WAL} = \frac{1}{\lambda} = \frac{1}{0.0989} = 10.11\,\text{years} \tag{23}$$

References

[1] Gary Schurman, *The Exponential Distribution - Modeling Arrival Times,* September, 2015.

[2] Gary Schurman, *Integration By Parts - Weighted-Average Revenue Life,* January, 2020.